Algorithms for Network Flows

Lecture 3: Generalized flows I

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Slides will be available at: `http://nolver.net/home/valparaiso`

Today's lecture

- A second path to a strongly polynomial algorithm for min cost flow
- The generalized flow model
- Building up our toolbox and intuition

Another route to a strongly polynomial algorithm

- The key to the strongly polynomial analysis was that

$$c^{\pi}(e) \geq 2n\epsilon(f) \quad \Rightarrow \quad e \notin E_{f^*} \text{ for any optimal } f^*.$$

$$c^{\pi}(e) \geq -\epsilon(f) \quad \forall e \in E_f$$

Another route to a strongly polynomial algorithm

- The key to the strongly polynomial analysis was that

$$c^\pi(e) \geq 2n\epsilon(f) \quad \Rightarrow \quad e \notin E_{f^*} \text{ for any optimal } f^*.$$

- There is a "dual" version of this. **We switch to the transshipment setting.**

> Call $e \in E$ contractible if $c^{\pi^*}(e) = 0$ for any optimal dual solution π^*.

- If we can prove that an edge e is contractible, we will be able to reduce the problem to a smaller instance.

Edge contraction

$$\min \sum_{e \in E} c(e) f(e) \qquad \max \sum_{i \in V} b_i \pi_i$$

$$\text{s.t.} \quad \nabla f_i = b_i \quad \forall i \in V \qquad \text{s.t.} \quad \pi_i - \pi_j \leq c(ij) \quad \forall ij \in E$$

$$f \geq 0$$

$$c^{\pi^*}(uv) = 0 \qquad \pi^*_u - \pi^*_v = c(uv)$$

So replace π_u by $\pi_v + c(uv)$

$$\max \sum_{i \in V \setminus \{u,v\}} b_i \pi_i + (b_u + b_v)\pi_v + b_u c(uv)$$

How can we show that an edge is contractible?

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Lemma

Let $f : E \to \mathbb{R}_+$ and $\pi : V \to \mathbb{R}$ be such that $c^\pi(e) \geq 0$ for all $e \in E_f$. Let

$$\text{Ex}(f) := \sum_{i \in V} \max\{\nabla f_i - b_i, 0\}.$$

If $f(\hat{e}) > \text{Ex}(f)$, then $\hat{e}$ is contractible.

Pf: Let f^* be an opt. flow, chosen s.t. $\|f - f^*\|_1$.

Let $h = f^* - f = \sum \lambda_i \chi(P_i)$

Claim: Each P_i is a path from

$$V^- = \{ i : \nabla f_i < b_i \}$$

to $$V^+ = \{ i : \nabla f_i > b_i \}$$

Pf: Suppose P_i is a cycle.

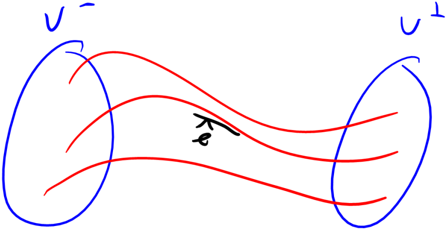
$$\text{supp}(h) \subseteq E_f, \quad \text{rev}(\text{supp}(h)) \subseteq E_{f^*}$$

π^* optimal dual sol^n.

$$c^{\pi}(P_i) \geq 0$$

$$\therefore c^{\pi^*}(P_i) \geq 0.$$

$\therefore \hat{f} = f^* - \delta \cdot P_i$ is "better" ↯. ☒.



$$h(\tilde{e}) \le E_x(f)$$

$$\therefore \; f^n(\tilde{e}) > 0.$$

Scheme of a strongly polymomial algorithm *Orlin '93*

FIND-OPTIMAL-DUAL(G):

1: Adjust f, π maintaining $c^\pi(e) \geq 0$ for all $e \in E_f$, to produce an edge e' with $f(e') > \mathrm{Ex}(f)$.
2: $\pi' \leftarrow$ FIND-OPTIMAL-DUAL($G/\{e'\}$)
3: "Uncontract" π' to get π^*

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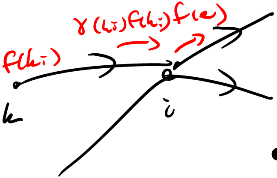
- Once an optimal dual π^* has been found, it's easy to find an optimal flow f^* by complementary slackness.

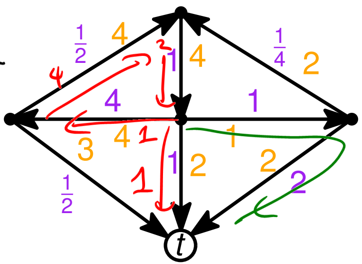
Generalized flow maximization

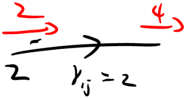
Given: Directed graph $G = (V, E)$, edge capacities $u : E \to \mathbb{R}_+$, gains $\gamma : E \to \mathbb{R}_{++}$, sink $t \in V$.
Goal: Find a generalized flow maximizing the net flow into t

- A generalized flow is a function $f : E \to \mathbb{R}_+$ with $f(e) \leq u(e)$ for all $e \in E$, and $\nabla f_i = 0$ for all $i \in V \setminus \{t\}$, where

$$\nabla f_i := \sum_{e \in \delta^-(i)} \gamma(e) f(e) - \sum_{e \in \delta^+(i)} f(e).$$



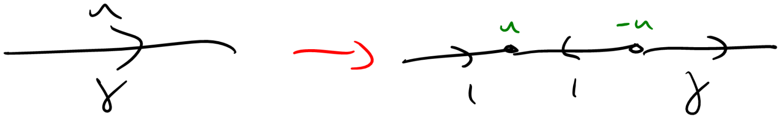




An equivalent formulation

- We can replace edge capacities by node demands $b : V \to \mathbb{R}$.

$$\begin{aligned} \max \quad & \nabla f_t \\ \text{s.t.} \quad & \nabla f_i = b_i \qquad \forall i \neq t \\ & f \geq 0 \end{aligned}$$



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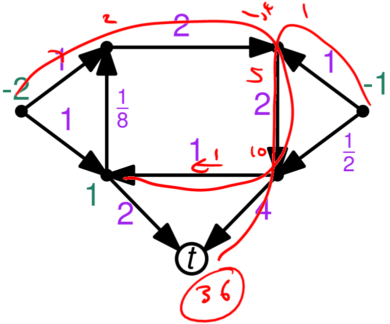
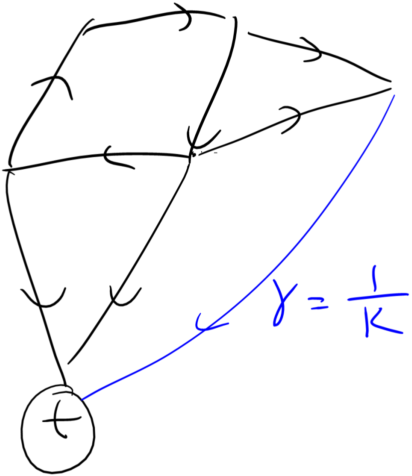


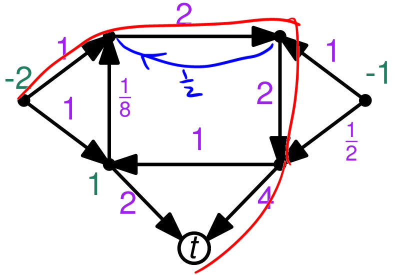
- **Extra assumption:** There is a path from i to t in E, for each $i \in V$.



- **Extra assumption:** There is a path from i to t in E, for each $i \in V$.
- We can allow flow to be discarded

$$\begin{aligned} \max \quad & \nabla f_t \\ \text{s.t.} \quad & f_i \geq b_i \qquad \forall i \neq t \\ & f \geq 0 \end{aligned}$$

Residual graph



- For $e \in E$, define $\gamma(\text{rev}(e)) = 1/\gamma(e)$.
- Given $f : E \to \mathbb{R}_+$, the residual capacity of an arc $e \in \overleftrightarrow{E}$ is

$$u_f(e) = \infty \qquad \forall e \in E$$
$$u_f(\text{rev}(e)) = f(e) \cdot \gamma(e) \qquad \forall e \in E$$

Generalized flow and LP

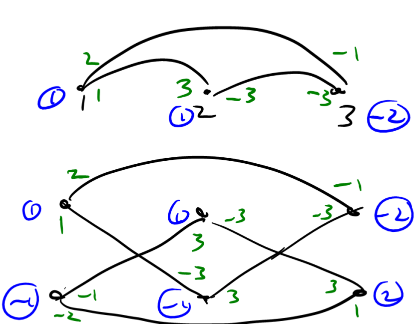
- Consider the feasibility problem

$$Ax = b, x \geq 0,$$

where $A \in \mathbb{R}^{mn}$ and each column of A has at most 2 nonzero entries.

> This is equivalent to the decision version of generalized flow maximization. *Hochbaum*

$$\begin{pmatrix} 1 & 2 & 0 \\ 3 & 0 & -3 \\ 0 & -1 & -3 \end{pmatrix} \underline{x} = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$



- What about the optimization LP

$$\min c^T x \qquad \text{s.t.} \qquad Ax = b, x \geq 0,$$

  same conditions on *A*?

- This is equivalent to the minimum cost generalized flow problem.

$$\min \sum c(e) f(e)$$

$$\nabla f_i = b_i$$

$$f \geq 0$$

- What about the optimization LP

$$\min c^T x \qquad \text{s.t.} \qquad Ax = b, x \geq 0,$$

  same conditions on A?

- This is equivalent to the minimum cost generalized flow problem.

- We don't know a strongly polynomial algorithm for this problem!

- Primal feasibility $\equiv$ max. generalized flow: Végh '14
- Dual feasibility $A^T y \leq c$: Megiddo '83

Flow-generating cycles

A cycle $C \in \overleftrightarrow{E}$ is called

- a flow-generating cycle if $\gamma(C) := \prod_{e \in C} \gamma(e) > 1$
- a flow-absorbing cycle if $\gamma(C) < 1$
- a unit cycle if $\gamma(C) = 1$

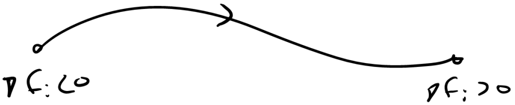
Flow decomposition

Let $f : E \to \mathbb{R}_+$ satisfy $f \leq u$. We say $g : E \to \mathbb{R}_+$ conforms to f if

$$\mathrm{supp}(g) \subseteq \mathrm{supp}(f)$$
$$\nabla g_i > 0 \Rightarrow \nabla f_i > 0$$
$$\nabla g_i < 0 \Rightarrow \nabla f_i < 0$$

Lemma

Let $f : E \to \mathbb{R}_+$ satisfy $f \leq u$. Then $f = \sum_{r=1}^{k} \lambda_r f^{(r)}$, where $k \leq m$, $\lambda \geq 0$, and each $f^{(r)}$ is an elementary flow conforming to f.



A trivial optimality condition

We call a network lossy if $\gamma(e) \leq 1$ for all $e \in E$.

Suppose the network is lossy, f is feasible with $\nabla f_i = b_i$ for all $i \neq t$, and $\gamma(e) = 1$ for all $e \in \text{supp}(f)$.
Then f is optimal.

Relabelling

- A labelling is any function $\mu : V \to \mathbb{R}_{++}$.

Given a labelling μ, define the relabelled gains γ^μ and relabelled demands b^μ by

$$\gamma^\mu_{ij} = \frac{\mu_i}{\mu_j} \cdot \gamma_{ij}, \qquad b^\mu_i = \frac{1}{\mu_i} \cdot b_i.$$

Given a flow f on the original instance, define the relabelled flow f^μ by

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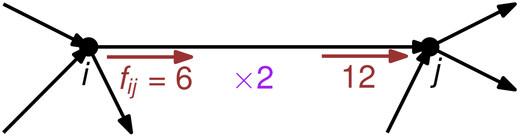

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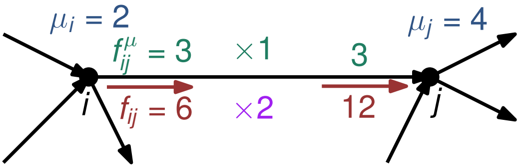
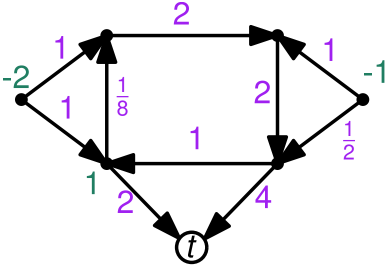
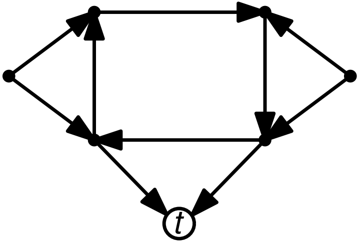


- Relabelled instance is completely equivalent to the original one





If $\nabla f_i = b_i$ forall $i \neq t$, and there exists a labelling μ s.t. G^μ is a lossy network with $\gamma^\mu(e) = 1$ for all $e \in \text{supp}(f^\mu)$, then f is optimal.

- Sufficient, but is it necessary?

If $\nabla f_i = b_i$ forall $i \neq t$, and there exists a labelling μ s.t. G^μ is a lossy network with $\gamma^\mu(e) = 1$ for all $e \in \text{supp}(f^\mu)$, then f is optimal.

- Sufficient, but is it necessary?

$$\begin{aligned} \min \quad & -z \\ \text{s.t.} \quad & \nabla f_i \geq b_i \quad \forall i \\ & \nabla f_t \geq z \\ & f \geq 0 \end{aligned}$$

Given $f \in \mathbb{R}_{+}^{E}$, $\mu \in \mathbb{R}_{++}^{V}$, (f, μ) is called a fitting pair if:

- μ is dual feasible
- $f_e > 0$ implies $\gamma_e^{\mu} = 1$.

- So if (f, μ) is a fitting pair and $\nabla f_i = b_i$ for all $i \neq t$, then f and μ are both optimal.
- Given a feasible f, there does not always exist a μ so that (f, μ) is a fitting pair...

Lemma

If there are no flow-generating cycles in E_f, then we can efficiently find labels μ s.t. $\gamma^{\mu}(e) \leq 1$ for all $e \in E_f$.

Cancelling flow generating cycles

Use a multiplicative version of Goldberg-Tarjan:

1: **while** $\exists$ a flow-generating cycle **do**
2: Find a cycle C in G_f of minimum mean gain $\gamma(C)^{1/|C|}$
3: Augment as much flow as possible around Γ

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- Weakly polynomial analysis is basically the same
- Strongly polynomial analysis is harder *Radzik '93*

Onaga's algorithm

1: Let f be an initial feasible solution ($\nabla f_i \geq b_i$ for all $i \neq t$)
2: Cancel all flow-generating cycles
3: **while** $\exists$ a node with $\nabla f_i > b_i$ **do**
4: Find a highest gain path P from i to t
5: Augment as much flow as possible via t

Lemma

After step 2, E_f never has any flow-generating cycles.

Lemma

Assuming rational input, Onaga's algorithm terminates with a maximum generalized flow.

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. . . But unfortunately this is not polynomial.

A weakly polynomial algorithm

Simpler version of an algorithm of Goldberg-Plotkin-Tardos ’91; see Wayne ’99, Shigeno ’04.

- Assume $b_i \in \mathbb{Z}$, $|b_i| \leq B$, and $\gamma(e) = \frac{p_e}{q_e}$ with $p_e, q_e \leq B$.

> A most improving path is a path in E_f that brings the largest amount of flow to the sink from a node with $\nabla f_i > b_i$.

Choose f satisfying $\nabla f_i = b_i$ for all $i \neq t$
repeat
    Cancel all flow-generating cycles
    Augment flow along a most improving path
until increase in ∇f_t in the iteration is less than B^{-2n}/m
Cancel all flow-generating cycles
Find a μ fitting f
μ will be an optimal dual solution; find f^* by complementary slackness.

Optimal duals to optimal primals

If μ is optimal, can compute an optimal g in strongly polynomial time.

Exercises

1. Explain how the generalized flow problem can be easily solved in strongly polynomial time if $b_i \leq 0$ for all $i \neq t$.

2. Suppose our generalized network has $b_i \in \mathbb{Z}$, $|b_i| \leq B$, and $\gamma(e) = \frac{p_e}{q_e}$ with $p_e, q_e \leq B$, where B is some integer. Assume $it \in E$ for each $i \neq t$.

   Suppose f is feasible ($\nabla f_i \geq b_i$ for all $i \neq t$), and that (f, μ) is a fitting pair. Prove that if

   $$\text{Ex}(f) := \sum_{i \neq t} (\nabla f_i - b_i) < B^{-3n},$$

   then there exists g with (g, μ) a fitting pair and $\nabla g_i = b_i$ for all $i \neq t$. (This implies that g and μ are both optimal.)

   Hint: work in the relabelled network with demands b_i^μ, gains $\gamma^\mu(e)$. Make use of integrality properties of regular (not generalized) flows.

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M. Shigeno.
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