Algorithms for Network Flows

Lecture 2: Minimum cost flows

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Slides will be available at: `http://nolver.net/home/valparaiso`

Minimum cost flow

Given: Directed graph $G = (V, E)$, edge capacities $u : E \to \mathbb{R}_+$, costs $c : E \to \mathbb{R}$, demands $b : V \to \mathbb{R}$ with $b(V) = 0$.
Goal: Find a b-flow of minimum cost.

- *b*-flow: function $f : E \to \mathbb{R}_+$ with $\nabla f_i = b_i$ for all $i \in V$, $f(e) \leq u(e)$ for all $e \in E$.
- The cost of f is $\sum_{e \in E} c(e) f(e)$.

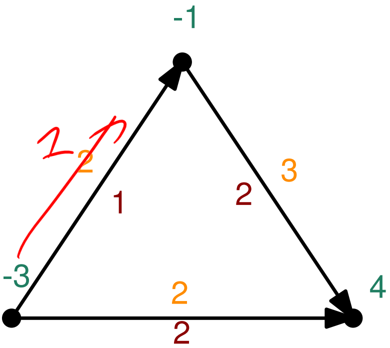
Today's lecture

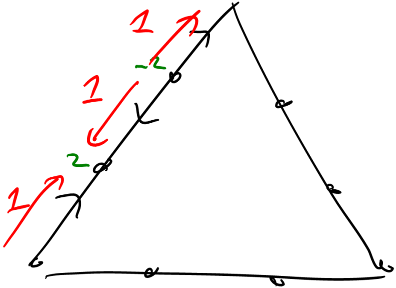
- A strongly polynomial time algorithm for min cost flow due to Goldberg-Tarjan '88.

  (The first strongly polynomial algorithm for the problem was by Tardos '85.)

Transshipment problem

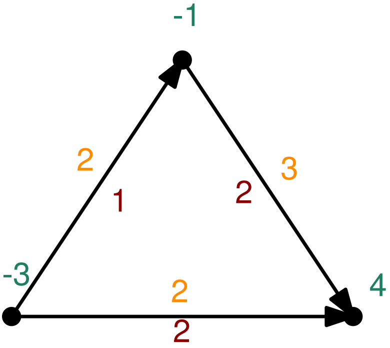
- Same, but with $u(e) = \infty$ for all e.
- Polynomially equivalent

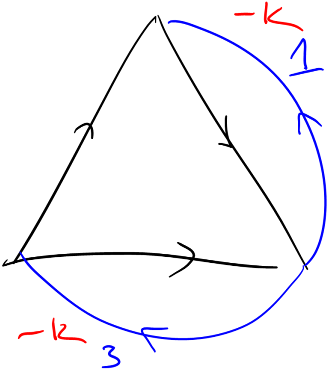




Min cost circulation - also equivalent

Given: Directed graph $G = (V, E)$, edge capacities $u : E \to \mathbb{R}_+$, costs $c : E \to \mathbb{R}$.
Goal: Find a circulation of minimum cost.





A trivial optimality condition

- If $c \geq 0$, then clearly $f = 0$ is the optimal circulation.
- Slightly more: if f is a circulation with $f(e) = 0$ whenever $c(e) > 0$ and $f(e) = u(e)$ whenever $c(e) < 0$, then f is optimal.

For $e \in \overleftrightarrow{E} \setminus E$, define $c(e) = -c(\mathrm{rev}(e))$

Then we can rewrite as simply:

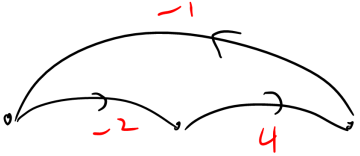
$$c(e) \geq 0 \quad \forall e \in E_f.$$

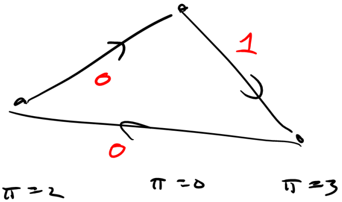
Relabelling

- A labelling (or potential) is any function $\pi : V \to \mathbb{R}$.

Given a labelling π, define the relabelled costs c^π by

$$c^\pi(ij) = c(ij) + \pi_i - \pi_j.$$



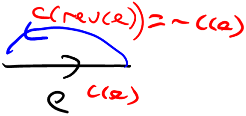


$c^\pi(\Gamma) = c(\Gamma)$

$\forall$ cycles Γ

$c^\pi(f) = c(f)$ for any circulation

Relabelling



- A labelling (or potential) is any function $\pi : V \to \mathbb{R}$.

> Given a labelling π, define the relabelled costs c^π by
>
> $$c^\pi(ij) = c(ij) + \pi_i - \pi_j.$$

Hence if $\exists\ \pi$ s.t. $c^\pi(e) \geq 0\ \forall e \in E_f$

Then f is optimal.

Relabelling

- A labelling (or potential) is any function $\pi : V \to \mathbb{R}$.

Given a labelling π, define the relabelled costs c^π by

$$c^\pi(ij) = c(ij) + \pi_i - \pi_j.$$

Clearly sufficient, is it necessary?

LP, dual and complementary slackness

$$\min \quad \sum_{e \in E} c(e) f(e)$$

$$\text{s.t.} \quad f(e) \leq u(e) \quad \forall e \in E \qquad \sigma_e$$

$$f(\delta^+(v)) - f(\delta^-(v)) = 0 \quad \forall v \in V \qquad \pi_i$$

$$f \geq 0$$

Dual:

$$\max \; -\sum u(e)\sigma_e$$

$$\text{s.t.} \; -\sigma_{ij} + \pi_j - \pi_i \leq c(ij) \quad \forall ij \in E$$

$$\sigma \geq 0$$

$$\Leftrightarrow \sigma_{ij} \geq -c^\pi(ij)$$

Best choice of σ_e

$$\max \; - \sum_{ij \in E} \max(-c^\pi(ij), 0)\, u(ij)$$

$$\text{s.t.} \quad \pi : V \to \mathbb{R}.$$

Equivalently,

$$\max \sum_{ij \in E} \min(c^\pi(ij), 0)\, u(ij)$$

$$\text{s.t.} \quad \pi : V \to \mathbb{R}$$

Comp. slackness:

$f(e) < u(e) \Rightarrow \alpha_e = 0 \Rightarrow c^\pi(ij) \geq 0$

$c^\pi(e) > 0 \Rightarrow f(e) = 0.$

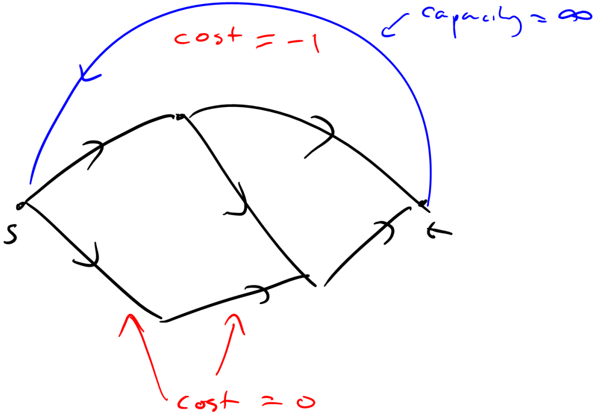
f optimal iff $\nexists$ cycle $\Gamma \subseteq E_f$ with $c(\Gamma) < 0$.

The Goldberg-Tarjan algorithm

$f \leftarrow 0$
while $\exists$ a negative cost cycle **do**
    Find a cycle Γ in G_f of minimum mean cost $c(\Gamma)/|\Gamma|$.
    Push δ := $\min_{e \in C}\{u_f(e)\}$ units of flow backwards around Γ

- Finding a minimum mean cost cycle can be done in time $O(mn)$ by dynamic programming.

Comparing to Edmonds-Karp for max flow



ϵ-optimality

A circulation f is ϵ-optimal if $\exists\pi$ s.t. $c^\pi(e) \geq -\epsilon$ for all $\epsilon \in E_f$.

If all costs are integers, and f is ϵ-optimal for some $\epsilon < 1/n$, then f is optimal.

Pf: For any cycle Γ in E_f,

$$c(\Gamma) = c^\pi(\Gamma) \geq -\varepsilon|c| > -1$$

≥ 0.

Let

$$\mu(f) := \min_{\Gamma \text{ a cycle in } G_f} \frac{1}{|\Gamma|} \cdot c(\Gamma)$$

$$\epsilon(f) := \min\{\epsilon : f \text{ is } \epsilon\text{-optimal}\}$$

$c^\pi(e) \geq -\varepsilon(f)$

Lemma

$\mu(f) = -\epsilon(f)$.

Lemma

$\epsilon(f) = -\mu(f)$ is decreasing throughout the algorithm.

A weakly polynomial bound

Lemma

Let f_r be the flow obtained after j iterations of the G-T algorithm. Then $\epsilon(f_{s+m}) \leq (1 - 1/n) \cdot \epsilon(f_s)$ for any s.

A strongly polynomial bound

We call an edge $e \in \overleftrightarrow{E}$ ϵ-frozen if $e \notin E_g$ for any ϵ-optimal circulation g.

Claim

If f is ϵ-optimal w.r.t. π, and $c^\pi(e) \leq -2n\epsilon$, then e is ϵ-frozen.

The algorithm terminates after $O(m^2 n \log n)$ iterations.

Overview

Max flow	Min cost flow	
Capacity scaling Ahuja-Orlin	**Capacity scaling + contraction** Orlin	
Shortest paths Edmonds-Karp	**Minimum mean cycle** Goldberg-Tarjan	
Push-relabel Goldberg-Tarjan		

State of the art

- **Fastest weakly polynomial algorithm:** $\tilde{O}(m\sqrt{n}\,\mathrm{polylog}\,U)$
  Lee-Sidford '13

- **Fastest strongly polynomial algorithm:**
  $O(m\log n(m + n\log n)) = \tilde{O}(m^2)$ Orlin '93

Exercise

Consider the following variation of the Goldberg-Tarjan algorithm:

$f \leftarrow 0$, $\pi \leftarrow 0$
repeat
    while There exists a cycle $\Gamma \subseteq E_f$ with $c^\pi(e) < 0$ for all $e \in \Gamma$ **do**
        Augment on Γ
    Update π so that $c^\pi(e) \geq -\epsilon(f)$ for all $e \in E_f$.
until $\epsilon(f) = 0$

Show that this runs in time $O(mn^2 \log(Cn))$ (so a factor m faster than what we got for the original Goldberg-Tarjan algorithm).

You may assume that the last step can be done in time $O(mn)$, and that a finding a cycle (if any) in a directed graph can be done in time $O(n)$.

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